

Solutions in the Large for Nonhomogeneous Quasilinear Hyperbolic Systems of Equations

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1. INTRODUCTION

We consider the Cauchy problem of the nonhomogeneous quasilinear hyperbolic system of equations

$$\frac{\partial v}{\partial t} + \frac{\partial f(v)}{\partial x} = g(v) e^{-\kappa t}, \quad -\infty < x < +\infty, t > 0, \quad (1.1)$$

$$v(x, 0) = v_0(x), \quad -\infty < x < +\infty. \quad (1.2)$$

Here, v , $f(v)$ and $g(v)$ are N -vector-valued functions; $f(v)$ and $g(v)$ are smooth of v . System (1.1) is assumed to be strictly hyperbolic and genuinely nonlinear in the sense of Lax [1], that is, the matrix $\partial f(v)/\partial v$ has real and distinct eigenvalues $\lambda_1(v) < \lambda_2(v) < \dots < \lambda_N(v)$ with corresponding right eigenvectors $\gamma_1(v), \gamma_2(v), \dots, \gamma_N(v)$ such that $\gamma_j \cdot \nabla \lambda_j \neq 0$ ($j = 1, \dots, N$). For definiteness, we always choose γ_j in a smooth manner so that $\gamma_j \cdot \nabla \lambda_j = 1$. The constant K is to be chosen later.

Since (1.1), (1.2) generally does not have smooth solutions, we look for weak solutions in the distributional sense; i.e., a bounded measurable function $v(x, t)$ is a solution if

$$\begin{aligned} & \iint_{t \geq 0} \left(v \frac{\partial \phi}{\partial t} + f(v) \frac{\partial \phi}{\partial x} + g(v) e^{-\kappa t} \phi \right) dx dt \\ & + \int_{t=0} v_0(x) \phi(x, 0) = 0 \end{aligned} \quad (1.3)$$

for any smooth vector-valued function ϕ with compact support in $t \geq 0$.

For the homogeneous system corresponding to (1.1), i.e., $g = 0$,

$$\frac{\partial v}{\partial t} + \frac{\partial f(v)}{\partial x} = 0. \quad (1.4)$$

Glimm [2] proved the existence theory of the solution of the Cauchy problem (1.4), (1.2) if the total variation of $v_0(x)$ is small. Such kinds of systems have been extensively studied, but in the nonhomogeneous case the result which has been found by us is the work of Tai-Ping Liu [3]. The system studied by Tai-Ping Liu and system (1.1) in this article have different kinds of nonhomogeneous terms.

2. THE DIFFERENCE SCHEME

The difference scheme which we use for solving the Cauchy problem (1.1), (1.2) is a generalization of Glimm's scheme. The block in Glimm's scheme is the solution $\tilde{v}(x, t, 0, 0, v_l, v_r)$ of the Riemann problem (1.4) with initial data

$$\begin{aligned}\tilde{v}(x, 0) &= v_l, & x < 0, \\ &= v_r, & x > 0.\end{aligned}$$

Following Lax [1] if v_r is close enough to v_l , the solution \tilde{v} exists and takes on constant values $v_0 = v_l, \dots, v_j, \dots, v_N = v_r$ in sectors $\dots, \sum_j = \{(x, t): \xi_j t < x < \xi_j' t\}, \dots$. The sectors \sum_{j-1} and \sum_j are consecutive. Also by [1] we have two possibilities for \tilde{v} in the sector between \sum_{j-1} and \sum_j .

(a_j) \sum_{j-1} and \sum_j have common boundary $x = \xi_j t$, also $\lambda_j(\tilde{v}(t\xi_j - 0, t)) > \xi_j > \lambda_j(\tilde{v}(t\xi_j + 0, t))$ and $v_j \in s_j(v_{j-1})$, where $s_j(v_{j-1}) = \{v: (v - v_{j-1})\xi_j = f(v) - f(v_{j-1}), \lambda(v) < \xi_j < \lambda(v_{j-1})\}$.

(b_j) Between \sum_{j-1} and \sum_j , $\lambda_j(v_{j-1}) \leq \lambda_j(\tilde{v}(t\xi, t)) < \lambda_j(v_j)$ and $\tilde{v}(t\xi, t) \in R_j(v_{j-1})$, where $R_j(v_{j-1}) = \{v: v \text{ is connected to } u_0 \text{ by an integral curve of } \gamma_j \text{ and } \lambda_j(v) > \lambda_j(v_{j-1})\}$.

In case (a_j) there is a jump discontinuity. This jump discontinuity is called a j -shock wave. In case (b_j), v is continuous on and between \sum_{j-1} and \sum_j . The configuration (b_j) is called a j -rarefaction wave, while a j -wave refers to either. $\tilde{v}(x, t, 0, 0, v_l, v_r)$ is called the resolution of the discontinuity v_l, v_r for system (1.4) at point $(0, 0)$ into j -waves, $1 \leq j \leq N$. v_0, v_1, \dots, v_N are called the intermediate states of \tilde{v} . Again by [1] they are c^2 functions of v_l and v_r and near v_l .

Now we describe the difference scheme we use for solving (1.1), (1.2). Randomly choose sequence $a = \{a_i, i \geq 1\}$, equidistributed in $(-1, 1)$, and take mesh length $\Delta x = l, \Delta t = h$ satisfying the Courant-Friedrich-Lewy condition

$$l/h = \text{const} \geq |\lambda_j(v)|, \quad j = 1, \dots, N,$$

for all v under consideration.

First we define $\tilde{v}(x, t, m+1, k, v_l, v_r) = \tilde{v}(x - (m+1)l, t - kh, 0, 0, v_l, v_r)$ as the resolution of the discontinuity v_l, v_r for system (1.4) at point $((m+1)l, kh)$. Then we take $u(x, t, m+1, k, v_l, v_r) = \tilde{v}(x, t, m+1, k, v_l, v_r) + g(\tilde{v}(x, t, m+1, \cdot, k, v_l, v_r)) (t - kh)e^{-Kt}$ as the approximate solution of system (1.1) with the initial data

$$\begin{aligned} u(x, kh) &= v_l, & x < (m+1)l, \\ &= v_r, & x > (m+1)l. \end{aligned}$$

Throughout this article we always assume the total variation over $(-\infty, +\infty)$ of $v_0(x)$, $\bigvee_{-\infty < x < +\infty} v_0(x)$ is small. Here and from now on for vector-valued function $v(x) = (v^{(1)}(x), \dots, v^{(N)}(x))$ we denote the total variation of $v(x)$ over an interval (a, b) as

$$\bigvee_{(a,b)} v(x) = \sup_{a < x_1 < \dots < x_N < b} \sum_{i=1}^{N-1} \|v(x_{i+1}) - v(x_i)\|,$$

where

$$\|v(x_{i+1}) - v(x_i)\| = \left(\sum_{j=1}^N (v^{(j)}(x_{i+1}) - v^{(j)}(x_i))^2 \right)^{1/2},$$

$x_1 \dots x_N$ are all possible sub-divisional points in (a, b) .

The construction of the difference approximation $u = u(x, t)$ proceeds as follows: Let

$$\begin{aligned} v_{0,l}(x) &= v_0(ml), & (m-1)l \leq x < (m+1)l \\ & & \text{and } |x| \leq (2[1/l]^2 + 1)l \\ &= v_0(-\infty), & (m-1)l \leq x < (m+1)l \\ & & \text{and } x < -(2[1/l]^2 + 1)l, m = \text{even}, \\ &= v_0(+\infty), & (m-1)l \leq x < (m+1)l \\ & & \text{and } x > (2[1/l]^2 + 1)l, \end{aligned}$$

then we define $u(x, t) = u(x, t, (m+1), 0, v_{0,l}(ml), v_{0,l}((m+2)l))$, for $ml \leq x < (m+2)l$, $0 < t < h$, $m = \text{even}$. Inductively suppose $u(x, t)$ exists for $0 \leq t < kh$; then we define

$$\begin{aligned} u(x, t) &= u(x, t, m+1, k, u((m+a_k)l, kh-0), \\ & \quad u((m+2+a_k)l, kh-0)), \end{aligned}$$

for

$$ml \leq x < (m+2)l, \quad kh \leq t < (k+1)h, \quad m+k = \text{even}.$$

Obviously, $u(x, t)$ depends on the mesh length h and sequence a ; we should express it by u_{ha} , but we may omit these suffixes for convenience.

Now we cannot simply show that the difference approximation u can be defined for $t > 0$. Instead, this will be proved simultaneously with the proof of the bounds of u under some restriction on the initial data $v_0(x)$ and the constant K .

3. ESTIMATES FOR THE DIFFERENCE EQUATION

Before we obtain our estimates on the difference approximation u , we consider the Cauchy problem of the system of the ordinary differential equations

$$\frac{dv}{dt} = g(v), \quad (3.1)$$

$$v|_{t=0} = v_0(-\infty).$$

Because of the smoothness of $g(v)$ it follows that there is a constant K' such that the Cauchy problem (3.1), possesses a unique bounded solution $v = \psi(\tau)$ on $[0, 1/K']$, i.e., $\|\psi(\tau)\| \leq M$.

It is not difficult to show that there are $D'_2 \subset D'_1$, where

$$D'_i = \left\{ (v, \tau): v \in D_i(\tau), \tau \in \left[0, \frac{1}{K'} \right] \right\};$$

$$D_i(\tau) = \{v: \|v - \psi(\tau)\| \leq \delta_i\}, \quad \tau \in [0, (1/K')],$$

such that the following properties are established.

In D'_1 there is a family of smooth coordinate functions $\omega_1(v, \tau), \dots, \omega_N(v, \tau)$, that is, $\gamma_i(v) \nabla \omega_i(v, \tau) = 1, \forall v \in D_1(\tau), \gamma_j(\psi(\tau)) \nabla \omega_i(\psi(\tau), \tau) = \delta_{i,j}$ and $\omega_i(\psi(\tau), \tau) = 0, \tau \in [0, 1/K'], i, j = 1, \dots, N$. There are the constants $M_1, M_2 > 1$ such that if v_A, v_B both belong to $D_1(\tau), \tau \in [0, 1/K']$, we have

$$\|v_A - v_B\|_\infty \leq M_1 \|v_A - v_B\|, \quad (3.2)$$

$$\|v_A - v_B\| \leq M_2 \|v_A - v_B\|_\infty, \quad (3.3)$$

where $\|v_A - v_B\|_\infty = \sup_{1 \leq j \leq N} |\omega_j(v_A, \tau) - \omega_j(v_B, \tau)|$. There exists a constant M_3 such that

$$\sup_{\substack{v \in D'_1 \\ 1 \leq i, j \leq N}} \left| \frac{\partial g^{(i)}}{\partial v^{(j)}} \right| \leq M_3, \quad (3.4)$$

$$\sup_{v \in D'_1} \|g(v)\| \leq M_3, \quad (3.5)$$

where $v^{(i)}$ and $g^{(i)}$ are the i th components of v and g , respectively.

If all the intermediate states $v_0 = v_l, v_1, \dots, v_N = v_r$ of the resolution $\tilde{v}(x, t, 0, 0, v_l, v_r)$ of the discontinuity v_l, v_r for system (1.4) at point $(0, 0)$ belong to $D_1(\tau)$, we let the quantity

$$\varepsilon_j(v_l, v_l - v_r, \tau) = \omega_j(v_j, \tau) - \omega(v_{j-1}, \tau)$$

denote the magnitude of the j -wave in \tilde{v} . Because $\omega_j(v, \tau)$ is smooth in D'_1 and also, by [1], $\varepsilon_j(v_l, v_l - v_r, \tau)$ is a c^2 function of $v_l, v_l - v_r$, there is a constant K_2 such that

$$\left| \frac{\partial \varepsilon_i}{\partial (v_l^{(j)} - v_r^{(j)})} \right| \leq K_2, \quad 1 \leq j \leq N, \quad (3.6)$$

$$\left| \frac{\partial^2 \varepsilon_i}{\partial v_l^{(j)} \partial (v_l^{(k)} - v_r^{(k)})} \right| \leq K_2, \quad 1 \leq j, k \leq N, \quad (3.7)$$

$$\left| \frac{\partial^2 \varepsilon_i}{\partial \tau \partial (v_l^{(k)} - v_r^{(k)})} \right| \leq K_2, \quad 1 \leq k \leq N, \quad (3.8)$$

provided $v_0 = v_l, v_1, \dots, v_{N-1}, v_N = v_r$ belongs to $D_1(\tau)$, $\tau \in [0, 1/K']$.

For any v_l, v_m, v_r belonging to $D_2(\tau)$, $\tau \in [0, 1/K']$, all corresponding intermediate states of $\tilde{v}(x, t, 0, 0, v_l, v_m)$, $v(x, t, 0, 0, v_m, v_r)$, $v(x, t, 0, 0, v_l, v_r)$ belong to $D_1(\tau)$, $\tau \in [0, 1/K']$; the magnitudes of their corresponding i -wave are respectively $\gamma_i, \delta_i, \varepsilon_i$; there is a constant K_1 such that

$$|\varepsilon_i| \leq |\gamma_i| + |\delta_i| + K_1 D(\gamma, \delta), \quad i = 1, 2, \dots, N, \quad (3.9)$$

where the definition of $D(\gamma, \delta)$ is the same as that in [2].

Throughout this article we take $K \geq \max[K', A(16N^2K_1 + N)]$, where $A = M_2 K_2 ((N^{1/2} + 1)M_3 + 1)$.

It is obvious that $v = \psi((1/K)(1 - e^{-Kt}))$, $t \in [0, +\infty)$ is the unique bounded solution of the Cauchy problem

$$\frac{dv}{dt} = g(v) e^{-Kt}, \quad (3.1)'$$

$$v|_{t=0} = v_0(-\infty).$$

Let $D_i = \{(v, \tau) : v \in D_i(\tau), \tau \in [0, 1/K]\}$, $i = 1, 2$, then (3.2), (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), (3.9), and other aforementioned properties about D'_1, D'_2 are obviously true.

From now on we defined the 1-1 relationship between τ and t by means of the equality $\tau = (1/K)(1 - e^{-Kt})$, $0 < t < +\infty$.

Let α, β be two waves in $\tilde{v}(x, t, m+1, k, u((m+a_k)l, kh-0), u((m+2+a_k)l, kh-0))$ which cross the line $t = (k + \frac{1}{2})h$, $m+k = \text{even}$, $k = 0, 1, 2, \dots$; if $k = 0$, let $u((m+a_0), 0h-0) = v_{0,l}(ml)$. For definiteness

suppose α is a j -wave and β is an i -wave. When $j \neq i$, we say α and β approach if the wave lying to the left on $t = (k + \frac{1}{2})h$ (toward $c = -\infty$) has the larger index. When $j = i$, we say α and β approach if $\alpha \neq \beta$ and if they are not both rarefaction waves.

Let α', β' be two waves in $\tilde{v}(x, t, m, k, u(m + a_k)l, kh - 0), u((m + 1)l, kh - 0)$ and $\tilde{v}(x, t, (m + 1), k, u(m + 1)l, kh - 0), u((m + 2 + a_k)l, kh - 0)$, which cross the line, $t = (k + \frac{1}{2})h, m + k = \text{even}, k = 1, 2, \dots$. We say α', β' approach in the same way as α, β .

Let $K_0 = 4NK_1$. For any $u(x, t)$ we define

$$F(kh + 0) = L(kh + 0) + K_0 Q(kh + 0),$$

where

$$L(kh + 0) = \sum \{|\alpha| : \alpha \text{ crosses } t = (k + \frac{1}{2})h\},$$

$$Q(kh + 0) = \sum \{|\alpha| |\beta| : \alpha \text{ and } \beta \text{ cross } t = (k + \frac{1}{2})h \text{ and approach}\},$$

$$k = 0, 1, 2, \dots$$

We also define

$$F(kh - 0) = L(kh - 0) + K_0 Q(kh - 0),$$

where

$$L(kh - 0) = \sum \{|\alpha'| : \alpha' \text{ crosses } t = (k + \frac{1}{2})h\},$$

$$Q(kh - 0) = \sum \{|\alpha'| |\beta'| : \alpha' \text{ and } \beta' \text{ cross } t = (k + \frac{1}{2})h \text{ and approach}\},$$

$$k = 1, 2, \dots$$

Because of (3.2), when the total variation of $v_0(x) \bigvee_{-\infty < x < +\infty} v_0(x)$ is small enough, there is a constant C depending only on M_1 such that

$$F(0) = F(0h + 0) \leq C \bigvee_{-\infty < x < +\infty} V_0(x), \quad (3.10.1)$$

$$C \bigvee_{-\infty < x < +\infty} v_0(x) < e^{e/(1-e)}. \quad (3.10.2)$$

THEOREM 3.1. *If $\bigvee_{-\infty < x < +\infty} v_0(x)$ is small enough such that (3.10.1) and (3.10.2) are satisfied and*

$$\delta = M_2 C e^{e/(e-1)} \bigvee_{-\infty < x < +\infty} v_0(x) \leq \min(\delta_2/3, 2^{-12} N^{-2} K_1^{-1}) \quad (3.10.3)$$

then when h is small enough, the difference approximation $u(x, t)$ of (1.1), (1.2) can be defined for all $t > 0$, and

$$\begin{aligned} F(nh + 0) &\leq F(0) e^{e/(e-1)}, \\ u(x, nh + 0) &\in D_4(\tau_n), \quad n = 1, 2, \dots, \\ \bigvee_{-\infty < x < +\infty} u(x, nh + 0) &\leq \delta, \end{aligned} \quad (3.11)$$

where $\tau_n = (1/K)(1 - e^{-Knh})$, $D_4(\tau) = \{v: \|v - \psi(\tau)\| \leq 2\delta\}$, $\tau \in [0, 1/K]$.

Proof. In the proof of this theorem we repeatedly employed the facts that $\|\psi(\tau_{n+1}) - \psi(\tau_n)\|$ uniformly converges to zero as h tends to zero and $u(-\infty, nh)$, as the value of the Euler's broken line of the integral curve $\psi(\tau)$, uniformly converges to $\psi(\tau_n)$ as h tends to zero. When $n=0$, (3.11) is obviously true.

Suppose $u(x, t)$ has been defined for $t \leq kh$ and (3.11) is true for any $n \leq k$. It follows from the smoothness of λ_j that there is an M^* such that

$$\sup_{\substack{|v| \leq M + 2\delta \\ j=1, \dots, N}} |\lambda_j(v)| < M^*.$$

Noting that $u(x, kh + 0) \in D_4(\tau_k)$, we conclude that $u(x, t)$ can be defined for $kh \leq t \leq (k+1)h$ provided the ratio of the mesh lengths $l/h = M^*$.

It is easily seen from simple calculations that

$$\bigvee_{-\infty < x < +\infty} u(x, (k+1)h - 0) \leq (1 + hNM_3) \bigvee_{-\infty < x < +\infty} u(x, kh + 0) \leq \frac{3}{2} \delta, \quad (3.12)$$

$$\begin{aligned} \|\psi(\tau_{k+1}) - u(x, (k+1)h - 0)\| &\leq \|\psi(\tau_{k+1}) - u(-\infty, (k+1)h)\| \\ &+ \bigvee_{-\infty < x < +\infty} u(x, (k+1)h - 0) \leq 2\delta, \end{aligned} \quad (3.13)$$

that is, $u(x, (k+1)h - 0) \in D_4(\tau_{k+1})$, provided h is small enough.

We define $D_3(\tau) = \{v: \|v - \psi(\tau)\| \leq 3\delta\}$, $\tau \in (0, 1/K)$; then because of (3.10.3), we obtain $D_4(\tau) \subset D_3(\tau) \subset D_2(\tau)$, $\tau \in [0, 1/K]$. In order to prove that (3.11) is true for $n = k+1$, we first prove a lemma.

LEMMA 3.1. *When h is small enough and v_l, v_r both belong to $D_4(\tau_k)$, $e_j(v_l, v_l - v_r, \tau_k)$ is the j -wave of $\tilde{v}(x, t, (m+1), k, v_l, v_r)$ with intermediate states v_0, v_1, \dots, v_N ($j = 1, 2, \dots, N$). Let $v'_l = v_l + g(v_l)he^{-K(k+1)h}$,*

$v'_r = v_r + g(v_r) h e^{-K(k+1)h}$, $\varepsilon'_j = \varepsilon_j(v'_l, v'_l - v'_r, \tau_{k+1})$ be the j -wave of $\tilde{v}(x, t, (m+1), k+1, v'_l, v'_r)$ ($j = 1, \dots, N$); then

$$|\varepsilon'_j - \varepsilon_j| < A h e^{-Kkh} \sum_{i=1}^N |\varepsilon_i|, \quad j = 1, 2, \dots, N, \quad (3.14)$$

where $A = M_2 K_2 ((N^{1/2} + 1) M_3 + 1)$.

Proof. When h is small enough, it follows from v_l, v_r both belonging to $D_4(\tau_k)$ that

$$\begin{aligned} \|\psi(\tau_{k+1}) - v'_l\| &\leq \|\psi(\tau_{k+1}) - \psi(\tau_k)\| + \|\psi(\tau_k) - v_l\| \\ &\quad + h e^{-K(k+1)h} \|g(v_l)\| \leq 3\delta, \end{aligned}$$

$$\begin{aligned} \|\psi(\tau_{k+1}) - v'_l - v_r + v_l\| &\leq \|\psi(\tau_{k+1}) - \psi(\tau_k)\| + \|\psi(\tau_k) - v_r\| \\ &\quad + h e^{-K(k+1)h} \|g(v_l)\| \leq 3\delta \end{aligned}$$

and $\|\psi(\tau_{k+1}) - v'_r\| \leq 3\delta$. These inequalities mean that v'_l, v'_r, v_l, v_r and $v_r - v_l + v'_l$ belong to $D_3(\tau_{k+1})$; that is, $\varepsilon_j(v_l, v'_l - v'_r, \tau_{k+1})$, $\varepsilon_j(v'_l, v_l - v_r, \tau_{k+1})$ and $\varepsilon_j(v_l, v_l - v_r, \tau_{k+1})$ can be defined in $D_1(\tau_{k+1})$.

Using $\varepsilon_j(v_l, 0, \tau) \equiv 0$ we obtain $(\partial \varepsilon_j / \partial \tau)(v_l, 0, \tau) = 0$, $(\partial \varepsilon_j / \partial v_l^{(i)})(v_l, 0, \tau) = 0$, $i, j = 1, 2, \dots, N$.

It follows from (3.6), (3.7) and (3.8) that

$$\begin{aligned} |\varepsilon'_j - \varepsilon_j| &\leq |\varepsilon_j(v'_l, v'_l - v'_r, \tau_{k+1}) - \varepsilon_j(v'_l, v_l - v_r, \tau_{k+1})| + |\varepsilon_j(v'_l, v_l - v_r, \tau_{k+1}) \\ &\quad - \varepsilon_j(v_l, v_l - v_r, \tau_{k+1})| + |\varepsilon_j(v_l, v_l - v_r, \tau_{k+1}) - \varepsilon_j(v_l, v_l - v_r, \tau_k)| \\ &\leq K_2 \|g(v_l) h e^{-K(k+1)h} - g(v_r) h e^{-K(k+1)h}\| \\ &\quad + K_2 \|v_l - v_r\| \|g(v_l)\| h e^{-K(k+1)h} + K_2 (\tau_{k+1} - \tau_k) \|v_l - v_r\| \leq K_2 h e^{-Kkh}. \end{aligned}$$

$$\begin{aligned} (N^{1/2} M_3 + M_3 + 1) \|v_l - v_r\| &\leq K_2 h e^{-Kkh} (N^{1/2} M_3 + M_3 + 1) \sum_{i=0}^{N-1} \|v_{i+1} - v_i\| \\ &\leq A h e^{-Kkh} \sum_{i=1}^N |\varepsilon_i|, \quad j = 1, 2, \dots, N. \end{aligned} \quad \text{Q.E.D.}$$

Now we continue the proof of this theorem. It is easy from Lemma 3.1 to obtain

$$L((k+1)h - 0) < (1 + N A h e^{-Kkh}) L(kh + 0). \quad (3.15)$$

Let α, β be two waves in $\tilde{v}(x, t, m+1, k, u((m+a_k)l, kh - 0), u((m+2+a_k)l, kh - 0))$ which cross the line $t = (k + \frac{1}{2})h$. Let α'', β'' be two waves in $\tilde{v}(x, t, m, k+1, u(ml, (k+1)h - 0), u((m+1+a_{k+1})l, (k+1)h - 0))$ and $\tilde{v}(x, t, m+1, k+1, u((m+1+a_{k+1})l, (k+1)h - 0), u((m+2)l, (k+1)h - 0))$ which cross the line $t = (k + 1 + \frac{1}{2})h$.

We say α corresponds to (each other) α'' if the two waves have the same index and they respectively belong to $\tilde{v}(x, t, m+1, k, u((m+a_k)l, kh-0), u)$ and $\tilde{v}(x, t, m, k+1, u(ml, (k+1)h-0), u((m+1+a_{k+1})l, (k+1)h-0))$ or they respectively belong to $\tilde{v}(x, t, m+1, k, u, u((m+2+a_k)l, kh-0))$ and $\tilde{v}(x, t, m+1, k+1, u((m+1+a_{k+1})l, (k+1)h-0), u((m+2)l, (k+1)h-0))$; u assumes one of the values of $u(x, kh+0)$ on $(ml, (m+2)l)$ and satisfies $u + g(u)he^{-Kt} = u(x, t, (m+1+a_{k+1})l, (k+1)h-0)$.

According to the definition of $Q((k+1)h-0)$ we have

$$\begin{aligned} Q((k+1)h-0) &= \sum \{|\alpha''| |\beta''| : \alpha'' \text{ and } \beta'' \text{ approach}\} \\ &= Q'((k+1)h-0) + Q''((k+1)h-0), \end{aligned}$$

where

$$\begin{aligned} Q'((k+1)h-0) &= \sum \{|\alpha''| |\beta''| : \alpha \text{ corresponding to } \alpha'' \text{ and } \beta \\ &\quad \text{corresponding to } \beta'' \text{ approach}\}, \\ Q''((k+1)h-0) &= \sum \{|\alpha''| |\beta''| : \alpha \text{ corresponding to } \alpha'' \text{ and } \beta \\ &\quad \text{corresponding to } \beta'' \text{ do not approach}\}. \end{aligned}$$

Using Lemma 3.1, we obtain

$$\begin{aligned} Q'((k+1)h-0) &\leq Q(kh+0) + 2NAhe^{-Kkh}L(kh+0)^2 \\ &\quad + N^2A^2h^2e^{-2Kkh}L(kh+0)^2. \end{aligned}$$

Note the fact that for any summand $|\alpha''| |\beta''|$ in $Q''((k+1)h-0)$, α'', β'' have the same index and at least one of them changes the sign of its magnitude from positive to negative. Using Lemma 3.1 and (3.15), we have

$$Q''((k+1)h-0) \leq NAhe^{-Kkh}(1 + NAhe^{-Kkh})L(kh+0)^2.$$

Summing up the above results, we obtain

$$\begin{aligned} F((k+1)h-0) &\leq (1 + NAhe^{-Kkh})F(kh+0) \\ &\quad + K_0(2NAhe^{-Kkh} + N^2A^2h^2e^{-2Kkh} \\ &\quad + NAhe^{-Kkh}(1 + NAhe^{-Kkh}))F(kh+0)^2. \end{aligned}$$

Because $F(nh+0) < F(0)e^{e/(e-1)} < 1$ for $n \leq k$ and $K_0(2NAe^{-Kkh} + hN^2Ae^{-2Kkh} + NAe^{-Kkh}(1 + NAhe^{-Kkh})) \leq 4NAK_0e^{-Kkh}$ as long as h is small enough, we have $F((k+1)h-0) \leq (1 + Khe^{-Kkh})F(kh+0)$.

By means of simple calculations we have $F((k+1)h-0) \leq F(0)e^{e/(e-1)}$.

It follows from (3.15) that $L((k+1)h-0) \leq 2\delta$ provided h is small enough. Noticing $u(x, (k+1)h-0) \in D_4(\tau_{k+1})$ and using the method similar to Glimm's for dealing with F_1 in [2] we conclude that

$$F((k+1)h+0) \leq F((k+1)h-0) \leq F(0) e^{e/(e-1)}. \quad (3.16)$$

Therefore,

$$\bigvee_{-\infty < x < +\infty} u(x, (k+1)h+0) \leq M_2 L((k+1)h+0) \leq M_2 F((k+1)h+0) \leq \delta, \quad (3.17)$$

$$\begin{aligned} \|\psi(\tau_{k+1}) - u(x, (k+1)h+0)\| &< \|\psi(\tau_{k+1}) - u(-\infty, (k+1)h)\| \\ &+ \bigvee_{-\infty < x < +\infty} u(x, (k+1)h+0) < 2\delta, \end{aligned} \quad (3.18)$$

that is, $u(x, (k+1)h+0) \in D_4(\tau_{k+1})$. Q.E.D.

For any point (x, t) which belongs to the half plane $t > 0$, there is n such that $nh \leq t < (n+1)h$. Using the same method as that used to get (3.12) and (3.13) we have

COROLLARY 3.1. *Under the same assumptions of Theorem 3.1, if h is small enough, for the difference approximation $u(x, t)$ of (1.1), (1.2) the following estimates are satisfied:*

$$\bigvee_{-\infty < x < +\infty} u(x, t) \leq 1\frac{1}{2}\delta, \quad (3.19)$$

$$\|u(x, t)\| \leq M + 2\delta. \quad (3.20)$$

LEMMA 3.2. *Under the same assumptions of Theorem 3.1, for any bounded integral $[-X, X]$, there is a constant D only depending on M^* , M_3 , δ and X such that*

$$\begin{aligned} I(X) = \int_{-X}^X \|u(x, t_2) - u(x, t_1)\| dx &\leq D(|t_2 - t_1| + h) \\ &\text{for } 0 < t_1, t_2 < +\infty. \end{aligned} \quad (3.21)$$

Proof. For definiteness suppose $t_1 < t_2$ and there are integers n, k such that $nh \leq t_1 < \dots < t_2 \leq (n+k+1)h$. It is obvious that

$$I(X) \leq I_1(X) + I_2(X), \quad (3.22)$$

where

$$\begin{aligned}
 I_1(X) &= \sum_{i=0}^{k+1} \int_{-X}^X \|u(x, (n+i)h+0) - u(x, (n+i)h-0)\| dx \\
 I_2(X) &= \sum_{i=0}^k \int_{-X}^X \|u(x, (n+i+1)h-0) - u(x, (n+i)h+0)\| dx \\
 &\quad + \int_{-X}^X (\|u(x, t_1) - u(x, nh+0)\| \\
 &\quad + \|u(x, t_2) - u(x, (n+k)h+0)\|) dx.
 \end{aligned}$$

It is not difficult to show that $I_1(X) \leq 4l[(t_2 - t_1)/h] + 3) \sup_{t>0} \bigvee_{-\infty < x < +\infty} u(x, t)$.

For any fixed x there exists an integer m such that $x \in [mL, (m+2)L]$; therefore

$$\begin{aligned}
 &\|u(x, (n+i+1)h-0) - u(x, (n+i)h+0)\| \\
 &\leq \bigvee_{(m-1)l \leq x \leq (m+3)l} u(x, (n+i)h+0) + M_3 h, \\
 &\|u(x, t_2) - u(x, (n+k)h+0)\| \\
 &\leq \bigvee_{(m-1)l \leq x \leq (m+3)l} u(x, (n+k)h+0) + M_3(t_2 - (n+k)h), \\
 &\|u(x, t_1) - u(x, nh+0)\| \\
 &\leq \bigvee_{(m-1)l \leq x \leq (m+3)l} u(x, nh+0) + M_3(t_1 - nh),
 \end{aligned}$$

so

$$I_2(X) \leq \left(\left\lceil \frac{t_2 - t_1}{h} \right\rceil + 4 \right) \left(4l \sup_{t>0} \bigvee_{-\infty < x < +\infty} u(x, t) + 2XM_3 h \right).$$

It follows from (3.19), (3.20) and (3.22) that $I(X) \leq (12\delta M^* + 2M_3 X)(t_2 - t_1) + (32M^* + 8XM_3)h$.

Let $D = \max((12\delta M^* + 2M_3 X), (32M^* + 8XM_3))$. Q.E.D.

4. CONVERGENCE OF THE APPROXIMATE SOLUTIONS

Because $u(x, t)$ is no longer an exact solution of (1.1) in the strip $kh < t < (k+1)h$, $k = 0, 1, 2, \dots$, in order to prove the existence of the weak solution $v(x, t)$ of the Cauchy problem (1.1), (1.2), we must first prove two lemmas.

LEMMA 4.1. *Under the same assumptions of Theorem 3.1, if $x' = \xi_j t'$ is a j -shock wave of $v(x, t, m+1, k, u((m+a_k)l, kh-0))$, $u((m+2+a_k)l, kh-0)$ in the domain $\{x, t: ml \leq x \leq (m+2)l, kh \leq t < (k+1)h, m+k = \text{even}\}$, then*

$$|\xi_j[u^{(n)}] - [f^{(n)}(u)]| < Wt' \|v_j - v_{j-1}\|, \quad n = 1, \dots, N, \quad (4.1)$$

where W is a constant depending on $f, g, v_0(x)$ only, $u^{(n)}$ and $f^{(n)}(u)$ are respectively the n th component of u and $f(u)$, $x' = x - (m+1)l$, $t' = t - kh$, v_0, v_1, \dots, v_N are the intermediate states of $\tilde{v}(x, t, m+1, k, u((m+a_k)l, kh-0), u((m+2+a_k)l, kh-0))$, $[\cdot]$ denotes the jump of the quantity in the bracket across the j -shock wave of \tilde{v} .

Proof. It follows from the smoothness of f and g and $|\xi_j| \leq M^*$ that

$$\begin{aligned} & |\xi_j[u^{(n)}] - [f^{(n)}(u)]| \\ &= |\xi_j(v_j^{(n)} + g^{(n)}(v_j) e^{-Kt't'} - v_{j-1}^{(n)} - g^{(n)}(v_{j-1}) e^{-Kt't'}) \\ &\quad - (f^{(n)}(v_j + g(v_j) e^{-Kt't'}) - f^{(n)}(v_{j-1} + g(v_{j-1}) e^{-Kt't'}))| \\ &= |\xi_j(g^{(n)}(v_j) - g^{(n)}(v_{j-1})) e^{-Kt't'} \\ &\quad - (f^{(n)}(v_{j-1}) - f^{(n)}(v_{j-1} + g(v_{j-1}) e^{-Kt't'})) \\ &\quad + (f^{(n)}(v_j) - f^{(n)}(v_j + g(v_j) e^{-Kt't'}))| \\ &= |\xi_j(v_j - v_{j-1}) \nabla g^{(n)}(v_*) \cdot e^{-Kt't'} \\ &\quad + \int_0^1 e^{-Kt't'} g(v_{j-1}) \nabla f^{(n)}(v_{j-1} + g(v_{j-1}) \eta e^{-Kt't'}) d\eta \\ &\quad - \int_0^1 e^{-Kt't'} g(v_{j-1}) \nabla f^{(n)}(v_j + g(v_j) \eta e^{-Kt't'}) d\eta| \\ &\leq \left(\left| \xi_j \frac{v_j - v_{j-1}}{\|v_j - v_{j-1}\|} \nabla g^{(n)}(v_*) \right| \right. \\ &\quad \left. + \int_0^1 \frac{v_j - v_{j-1}}{\|v_j - v_{j-1}\|} \nabla (g(v_{**}) \nabla f^{(n)}(v_{**} + g(v_{**}) \eta e^{-Kt't'})) d\eta \right) \|v_j - v_{j-1}\| t' \\ &\leq Wt' \|v_j - v_{j-1}\|, \end{aligned}$$

where v_* and v_{**} are points on the segment $\theta v_j + (1-\theta)v_{j-1}$, $0 \leq \theta \leq 1$.
 $n = 1, 2, \dots, N$. Q.E.D.

LEMMA 4.2. *Under the same assumptions of Theorem 3.1, if \tilde{v} is a j -rarefaction wave in the subdomain $\Omega_j = \{x, t: \xi_{j-1} < x'/t' < \xi_j, 0 \leq t' < h\}$ of the domain $\{x, t: ml \leq x < (m+2)l, kh \leq (k+1)h, m+k = \text{even}\}$; then*

$$\left| \int_{\Omega_j} (u\phi_t + f(u)\phi_x + e^{-\kappa t} g(u)\phi) dx dt + \int_{\partial\Omega_j} u\phi dx - f(u)\phi dt \right|$$

$$\leq R_1 h^3 + R_2 h^3 + R_2 h^2 \cdot \bigvee_{\substack{l_{j-1}t' < x' < l_j t' \\ t' = h/2}} \tilde{v}, \quad (4.2)$$

where $x' = x - (m+1)l$, $t' = t - kh$, $\tilde{v} = \tilde{v}(x, t, (m+1), k, u((m+a_k)l, kh+0), u((m+2+a_k)l, kh+0))$, ϕ is a smooth vector-valued function with compact support $t \geq 0$ and R_1 and R_2 are constants depending on ϕ , g , f and $v_0(x)$ only.

Proof. \tilde{v} , as a j -rarefaction wave in Ω_j , satisfies (1.4) in the classic sense in Ω_j . Noticing

$$\frac{\partial u}{\partial t} = \frac{\partial \tilde{v}}{\partial t} + t' g'(\tilde{v}) \frac{\partial \tilde{v}}{\partial t} e^{-\kappa t} + g(\tilde{v}) e^{-\kappa t} - t' K g(\tilde{v}) e^{-\kappa t},$$

$$\frac{\partial u}{\partial x} = \frac{\partial \tilde{v}}{\partial x} + t' g'(v) \frac{\partial \tilde{v}}{\partial x} e^{-\kappa t}$$

and letting $J(x, t) = \partial u / \partial t + f'(u)(\partial u / \partial x) - g(u)e^{-\kappa t}$, we get

$$\begin{aligned} J(x, t) &= \left(\frac{\partial \tilde{v}}{\partial t} + f'(u) \frac{\partial \tilde{v}}{\partial x} \right) - t' K g(\tilde{v}) e^{-\kappa t} \\ &\quad + (g(\tilde{v}) - g(u)) e^{-\kappa t} + e^{-\kappa t} t' (g'(\tilde{v}) \frac{\partial \tilde{v}}{\partial t} + f'(u) g'(\tilde{v}) \frac{\partial \tilde{v}}{\partial x}) \\ &= (f'(u) - f'(v)) \frac{\partial \tilde{v}}{\partial x} + t' e^{-\kappa t} (f'(u) g'(\tilde{v}) - g'(\tilde{v}) f'(\tilde{v})) \frac{\partial \tilde{v}}{\partial x} \\ &\quad - t' K g(\tilde{v}) e^{-\kappa t} + (g(\tilde{v}) - g(u)) e^{-\kappa t} \end{aligned}$$

and

$$\left| \iint_{\Omega_j} (u\phi_t + f(u)\phi_x + e^{-\kappa t} g(u)\phi) dx dt + \int_{\partial\Omega_j} u\phi dx - f(u)\phi dt \right|$$

$$\leq \left| \iint_{\Omega_j} J(x, t) dx dt \right| < J_1 + J_2.$$

Here

$$J_1 = \iint_{\Omega_j} R'_1 |\phi| t' dx dt \leq R_1 h^3,$$

$$J_2 = \iint_{\Omega_j} R'_2 |\phi| \left| \frac{\partial \tilde{v}}{\partial x} \right| t' dx dt$$

$$\leq R_2 h^2 \bigvee_{\substack{l_{j-1}t' < x' < l_j t' \\ t' = h/2}} \tilde{v},$$

where R'_1 and R'_2 are constants depending on g , f and $v_0(x)$ only. Q.E.D.

COROLLARY 4.1. *Under the same assumptions of Theorem 3.1, v_j is an intermediate state of \tilde{v} in the subdomain $\sum_j = \{x, t: \xi_j t' < x' < \xi' t', 0 \leq t' < h\}$ of the domain $\{(x, t): mL \leq x \leq (m+2)L, kh < t < (k+1)h, m+k = \text{even}\}$; then*

$$\left| \int_{\sum_j} u \phi_t + f(u) \phi_x + g(u) \phi \, dx \, dt + \int_{\partial \sum_j} u \phi \, dx - f(u) \phi \, dt \right| \leq R_1 h^3, \quad (4.3)$$

where x', t', v, ϕ and R_1 have the same meaning as in Lemma 4.2.

Now we are ready for proving the existence theorem.

THEOREM 4.1. *Suppose system (1.1) is strictly hyperbolic and genuinely nonlinear, f and g are smooth functions of v and $K \geq \max(K', A(16N^2K_1 + N))$. If initial data $v_0(x)$ are given to satisfy conditions (3.10.1), (3.10.2) and (3.10.3), then there is a weak solution $v(x, t)$ of Cauchy problem (1.1), (1.2) defined for all $t \geq 0$.*

Proof. We employ the difference scheme introduced in Section 2. We keep the ratio of the mesh lengths $l_i/h_i = M^*$, $i = 1, 2, \dots$, and let h_i tend to zero as i tends to $+\infty$. It follows from the above discussions that when i is large enough, the difference approximations $u_{hia}(x, t)$ (the suffixes h_i and a mean that $u_{hia}(x, t)$ depend on the mesh length h_i and sequence a) can be defined for all $t \geq 0$.

It follows from Lemmas 4.1 and 4.2 that for any given smooth vector-valued function ϕ with compact support $t \geq 0$ there are constants W, R_1 and R_2 which depend on ϕ, g, f and $v_0(x)$ only such that

$$\begin{aligned} |Q_{hia}| &= \left| \int_0^\infty \int_{-\infty}^{+\infty} (u_{hia} \phi_t + f(u_{hia}) \phi_x + g(u_{hia}) e^{-Kt} \phi) \, dx \, dt \right. \\ &\quad \left. + \int_{-\infty}^{+\infty} v_0, l_i(x) \phi(x, 0) \, dx \right| \\ &\leq \left| \sum_{n=1}^\infty \int_{-\infty}^{+\infty} \phi(x, nh_i) (u(x, nh_i - 0) - u(x, nh_i + 0)) \, dx \right| \\ &\quad + \frac{(2N+1)Y}{h_i^2} R_1 h_i^3 + \frac{T}{h_i} \left(R_2 + \frac{N^{1/2}W}{2} \right) h_i^2 (1\frac{1}{2}\delta), \end{aligned}$$

where Y is a constant concerned with the measure of the support of ϕ and T is a constant such that $\phi \equiv 0$ for $t \geq T$. Following Glimm [2], $|Q_{hia}|$ tends to zero as i tends to $+\infty$ for almost all randomly chosen sequences $a = \{a_i\}$, equidistributed in $(-1, 1)$. For any sequence a such that $|Q_{hia}|$ tends to zero as i tends to $+\infty$, we consider the corresponding difference approximations

$u_{h_{ik}a}$. It is easily seen from (3.19), (3.20), (3.21) that there is a subsequence $u_{h_{ik}a}$, $k = 1, 2, \dots$, such that $u_{h_{ik}a}$ converges to a bounded measurable $v(x, t)$ (uniformly for bounded t) on the intervals $|x| \leq X$ of any horizontal line. It is easy to see that $v(x, t)$ is the weak solution of the Cauchy problem (1.1), (1.2). Q.E.D.

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